MATH 579: Combinatorics
Exam 2 Solutions

1. Find the number of solutions in integers to $x_1 + x_2 + x_3 = 10$, with each $x_i \geq 3i - 5$.
The conditions are $x_1 \geq -2, x_2 \geq 1, x_3 \geq 4$. Set $y_1 = x_1 + 2, y_2 = x_2 - 1, y_3 = x_3 - 4$; having all of the $y_i \geq 0$ corresponds to our three conditions. We substitute to get $(y_1-2)+(y_2+1)+(y_3+4) = 10$ or $y_1 + y_2 + y_3 = 7$. This has $\left(\!\binom{3}{7}\!\right) = \binom{9}{7} = \binom{9}{2} = \frac{9^{\underline{2}}}{2!} = 36$ solutions. (or $\binom{9}{2} = \frac{9!}{7!2!} = \frac{9\cdot 8}{2!} = 36$)

2. Calculate $B(5)$ using only its recurrence relation and $B(0) = 1$.
We must repeatedly use the recurrence $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B(k)$. In turn, we calculate: $B(1) = \binom{0}{0}B(0) = 1$. $B(2) = \binom{1}{0}B(0) + \binom{1}{1}B(1) = 1 + 1 = 2$. $B(3) = \binom{2}{0}B(0) + \binom{2}{1}B(1) + \binom{2}{2}B(2) = 1 + 2 \cdot 1 + 1 \cdot 2 = 5$. $B(4) = \binom{3}{0}B(0) + \binom{3}{1}B(1) + \binom{3}{2}B(2) + \binom{3}{3}B(3) = 1 + 3 \cdot 1 + 3 \cdot 2 + 1 \cdot 5 = 15$. Finally, $B(5) = \binom{4}{0}B(0) + \binom{4}{1}B(1) + \binom{4}{2}B(2) + \binom{4}{3}B(3) + \binom{4}{4}B(4) = 1 + 4\cdot 1 + 6 \cdot 2 + 4 \cdot 5 + 1 \cdot 15 = 52$.

3. Prove that $S(n, n-2) = \frac{1}{24}n(n-1)(n-2)(3n-5)$, for all $n \geq 4$.
$S(n, n-2)$ counts partitions of $[n]$ into $n-2$ parts. These come in two types: (A) there is a part of size 3 (and the rest are of size 1); and (B) there are two parts of size 2 (and the rest are of size 1). For type A, there are $\binom{n}{3}$ ways to pick the big part, and the other parts are determined perforce. For type B, there are $\binom{n}{2}$ ways to pick one of the special parts, and $\binom{n-2}{2}$ ways to pick the other. However, this double-counts, as $\{1,2\}\{3,4\}$ is the same as $\{3,4\}\{1,2\}$. Hence, there are $\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$ partitions of type B. Hence $S(n, n-2) = \binom{n}{3} + \frac{1}{2}\binom{n}{2}\binom{n-2}{2} = \frac{1}{6}n(n-1)(n-2) + \frac{1}{8}n(n-1)(n-2)(n-3) = n(n-1)(n-2)\left(\frac{1}{6} + \frac{n-3}{8}\right) = n(n-1)(n-2)\frac{4+3(n-3)}{24} = n(n-1)(n-2)\frac{3n-5}{24}$.

4. Prove that the number of integer partitions of n into at most k parts, is equal to the number of integer partitions of n into any number of parts, each not larger than k.
The first quantity is the number of integer partitions whose Ferrers diagram has at most k rows. The second quantity is the number of integer partitions whose Ferrers diagram has at most k columns. Conjugation is a bijection between integer partitions counted by the two quantities of interest, because it swaps rows with columns.

5. Find all self-conjugate integer partitions of 23.
It's easier to find all integer partitions into distinct odd numbers, and then go backward to self-conjugate integer partitions. There must be an odd number of odd summands (since 23 is odd), and it can't be 5 (since $1+3+5+7+9 = 25 > 23$). Hence the possibilities are $23, 19+3+1, 17+5+1, 15+7+1, 15+5+3, 13+9+1, 13+7+3, 11+9+3, 11+7+5$. Drawing these as symmetric hooks, we find the self-conjugate partitions as, respectively: $12+1+1+1+1+1+1+1+1+1+1+1$, $10+3+3+1+1+1+1+1+1+1$, $9+4+3+2+1+1+1+1+1$, $8+5+3+2+2+1+1+1$, $8+4+4+3+1+1+1+1$, $7+6+3+2+2+2+1$, $7+5+4+3+2+1+1$, $6+6+4+3+2+2$, $6+5+5+3+3+1$.

6. Determine the number of surjective functions $f: N \to K$, where $|N| = n, |K| = k$, the elements of N are indistinct, and the elements of K are distinct. Be sure to justify your answer.
Such functions are bijective with multisets, drawn from K, of size n, where each element of K appears at least once (due to surjectivity). The bijection is given by the number of domain elements mapping to a particular codomain element. In turn, these are bijective with unrestricted multisets, drawn from K, of size $n-k$. The bijection is defined as removing exactly one copy of each element of K, e.g. $\{1^5 2^3 3^1\} \leftrightarrow \{1^4 2^2 3^0\}$. These latter have an established formula, namely $\left(\!\binom{k}{n-k}\!\right) = \binom{n-1}{n-k} = \binom{n-1}{k-1}$.